Killing-Yano Forms of a Class of Spherically Symmetric Space-Times I: A Unified Generation of Killing Vector Fields

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Abstract

Killing-Yano one forms (duals of Killing vector fields) of a class of spherically symmetric space-times characterized by four functions are derived in a unified and exhaustive way. For well-known space-times such as those of Minkowski, Schwarzschild, Reissner-Nordstrøm, Robertson-Walker and several forms of de Sitter, these forms arise as special cases in a natural way. Besides its two well-known forms, four more forms of de Sitter space-time are also established with ten independent Killing vector fields for which four different time evolution regimes can explicitly be specified by the symmetry requirement. A family of space-times in which metric characterizing functions are of the general form and admitting six or seven independent Killing vector fields is presented.

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I. INTRODUCTION

Defining relations of Killing-Yano (KY) and conformal KY-forms are natural generalizations of Killing 1-forms and conformal Killing 1-forms. The latter are the dual of Killing and conformal Killing vector fields whose flows generate, respectively, local isometries and local conformal isometries of the metric in (pseudo)Riemannian geometry [1, 2]. Although they are not related to the isometries of the metric these higher rank generalizations have attracted increasing interest in various fields of physics and modern mathematics as well as in some related fields. Generally speaking, many interesting properties of a space-time are intimately connected with the existence of (conformal) KY-forms admitted by the corresponding metric. More specifically; the determination of KY-forms of a given metric, classification of space-times admitting KY-forms, analysis of the algebraic structures of these forms as well as specification of the symmetry algebra and related conserved quantities of the Dirac and related equations in a given curved background have gained increasing significance. Equally important objects (not considered in this study) are the totally symmetric Killing tensors and their conformal generalizations (see [3, 4, 5] and references therein).

KY-forms play a prominent role in a unified description of null and non-null shear-free congruences [6] and in the search of force-free fields (divergence-free eigenvectors of the curl operator with position dependent eigenvalues) mostly encountered in astrophysics and fluid dynamics literatures [7, 8, 9]. A tangent vector which generates conformal transformations on its orthogonal complement is said to be the generator of shear-free congruence for its integral curve. Shear-free equation is a generalization of defining equation of conformal Killing equation. Together with Clifford calculus, KY-forms also provide efficient means in analyzing elliptic operators and the Dirac operator and in the further classification of (pseudo)Riemannian manifolds [1, 10]. While conformal KY-forms take part in symmetry operators for the massless Dirac equation [7, 11], KY-forms are indispensable in constructing first order symmetries of the massless as well as massive Dirac equation in a curved space-time [12]. KY-forms are also necessary for the symmetries of the Kähler equation [1, 13].

Longstanding interest in KY-forms largely stems from their constant use in general relativity and especially from their role in constructing conserved quantities in a number of ways. The studies of Penrose and his collaborators [14, 15], who have shown how the existence of a KY 2-form explains Carter's result on the integrability of the geodesic equation in Kerr

background, constitute a stepping-stone in this context [16, 17]. The fact that any KY p-form provides a quadratic first integral of the geodesic equation is by now a well-established particular result of the fact that the interior derivative of any KY p-form with respect to the tangent vector field of any geodesic remains parallel along the geodesic. More generally, any KY (p+1)-form is associated with a symmetric bilinear form, which is nothing more than the Killing tensor generalizing the so-called Stackel-Killing tensor that corresponds to a KY 2-form as first recognized by Penrose and Floyd. For more in this context, refer to [10].

On the other hand, as every KY form is co-closed, the Hodge dual of any KY p-form is directly associated with a conserved quantity. In the case of KY 1-forms, two types of conserved currents can be defined. For Ricci-flat space-times the Hodge dual of the exterior derivative of a KY 1-form ω is conserved, where ${}^*d\omega$ is known as the Komar form [1]. Secondly, the current $j = i_{X^a}\omega \wedge {}^{*^{-1}} G_a$ defined in terms of the Einstein 3-forms G_a, and KY 1-form ω is also conserved. Here, * and ${}^{*^{-1}}$ represent the Hodge map and its inverse, $\wedge$ denotes exterior multiplication, d and i_X stand for exterior and interior derivatives (with respect to vector field X). It has recently been shown how the KY-forms of the flat space-time can be used to construct new, conserved gravitational charges for transverse asymptotically flat [18] as well as for asymptotically anti de Sitter space-times [19]. These studies present another way of constructing a conserved current by taking a particular linear combination of wedge products of interior derivatives of the KY p-form and curvature characteristics of the underlying manifold. For KY 1-forms, this reduces to the usual current obtained from the Einstein 3-form.

The main purpose of this and the accompanying paper [20] is to develop, by directly starting from the KY-equation,

$$\nabla_{X_a}\omega_{(p)} = \frac{1}{p+1} i_{X_a} d\omega_{(p)} \ , \quad p = 1, 2, 3 \tag{1}$$

a constructive method which makes it possible to generate all KY forms for a large class of spherically symmetric space-times in a unified and exhaustive way. Here ∇_X stands for the covariant derivative with respect to the vector field X. It should be noted that the KY 0-form $\omega_{(0)}$ can be any function, $\omega_{(1)}$ is the dual of a Killing vector field and $\omega_{(n)}$ is a constant (parallel), that is, it is a constant multiple $\omega_{(n)} = az$ of the volume form z (for the orthonormal frame given below $z = e^{0123}$). The goal of our study is achieved by solving a coupled set of first order partial linear differential equations for the component functions of

the KY forms $\omega_{(p)}$ in each case. The number of independent equations that have to be solved for $p = 1, 2, 3$ are 10, 18 and 16, respectively. To obtain the most general solutions in an exhaustive way, there are also $24, 36$ and 24 integrability conditions that must be carefully examined at the outset. We first solve a suitable set of the integrability conditions by which the set of solutions naturally branches into cases and subcases.

We shall mainly use the notation of [1] and adopt the following conventions and terminology. The underlying base manifold is supposed to be a 4-dimensional ($4D$) pseudo-Riemannian manifold with the metric tensor g having Lorentzian signature $(-+++)$ such that $g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3$ in a local orthonormal co-frame $\{e^a\}$. By choosing the co-frame basis

$$\begin{aligned} e^0 &= H_0 dt \, , \qquad e^1 = TH_1 dr \, , \\ e^2 &= TH_2 d\theta \, , \quad e^3 = TH_2 \sin\theta d\varphi \, , \end{aligned}$$

a class of spherically symmetric metrics that will be considered can be parameterized by

$$T = \exp(\lambda(t)) \, , \quad H_j = H_j(r) \, ; \quad j = 0, 1, 2$$

which henceforth will be referred to as the (metric) coefficient functions. Here (t, r, θ, φ) specifies a local polar space-time chart with the usual range of variations. Whenever necessary, the range of r can be bounded to keep the coefficient functions real. A generic property of this kind of metric is invariance under the transformation of the spatial rotation group $SO(3)$. If g admits a time-like Killing vector field K_0, it is termed stationary and if, in addition, K_0 is orthogonal to a family of space-like hypersurfaces it is termed static. The dual tangent frame basis $\{X_a\}$ of the co-frame basis $\{e^a\}$ are,

$$\begin{aligned} X_0 &= \frac{1}{H_0}\partial_t \, , \qquad X_1 = \frac{1}{TH_1}\partial_r \, , \\ X_2 &= \frac{1}{TH_2}\partial_\theta \, , \quad X_3 = \frac{1}{TH_2 \sin\theta}\partial_\varphi \, , \end{aligned}$$

where $e^a(X_b) = \delta^a_b$ and $\partial_x = \partial/\partial x$. The metric dual of a vector field X will be denoted by $\tilde{X}$ such that $\tilde{X}(Y) = g(X, Y)$ for any vector field Y. The torsion-free connection 1-forms ω_{ab} for this class of metrics are well-known, and can be presented in the following antisymmetric

matrix-valued 1−form:

$$
(\omega_{ab}) = \frac{1}{T}\begin{pmatrix} 0 & & & \\ \frac{h_0}{H_1}e^0 + \frac{\dot{T}}{H_0}e^1 & 0 & & \\ \frac{\dot{T}}{H_0}e^2 & \frac{h_2}{H_1}e^2 & 0 & \\ \frac{\dot{T}}{H_0}e^3 & \frac{h_2}{H_1}e^3 & \frac{\cot\theta}{H_2}e^3 & 0 \end{pmatrix}, \tag{2}
$$

where we have used the abbreviations $\dot{T} = dT/dt$, $dH(r)/dr = H'$ and $h_j = H'_j/H_j$. We shall always use prime and over-dot to denote, respectively, the r-derivation and t-derivation of a function which depends only on r and t. The partial derivative of a map U of several variables with respect to x will be denoted by U_x, and of a rational map or function U/V by $\partial_x(U/V)$. The following matrix-valued 1-form is helpful in carrying out the calculations

$$
(\nabla_{X_a}e^b) = -\frac{1}{T}\begin{pmatrix} \frac{h_0}{H_1}e^1 & \frac{h_0}{H_1}e^0 & 0 & 0 \\ \frac{\dot{T}}{H_0}e^1 & \frac{\dot{T}}{H_0}e^0 & 0 & 0 \\ \frac{\dot{T}}{H_0}e^2 & -\frac{h_2}{H_1}e^2 & \frac{\dot{T}}{H_0}e^0 + \frac{h_2}{H_1}e^1 & 0 \\ \frac{\dot{T}}{H_0}e^3 & -\frac{h_2}{H_1}e^3 & -\frac{\cot\theta}{H_2}e^3 & \frac{\dot{T}}{H_0}e^0 + \frac{h_2}{H_1}e^1 + \frac{\cot\theta}{H_2}e^2 \end{pmatrix}. \tag{3}
$$

The corresponding curvature 2-forms are presented in Appendix B.

All the well-known spherically symmetric space-times such as the Minkowski, Schwarzschild, Reissner-Nordstrøm, Robertson-Walker and the six various forms of de Sitter models fall within the class of the considered metric as special cases and this provides the opportunity to give complete lists of their KY forms. As a particular result, we have found a completely solvable nonlinear ordinary differential equation $T^2\partial_t(\dot{T}/T) = \ell$, where ℓ is constant, characterizing five different time-dependent types of de Sitter space-time in a unified way. This fact also enables us to give explicit expressions of their KY-forms in a unified and exhaustive way. As the first part of our study, the present paper is entirely devoted to the unified generation of all Killing vector fields for the considered class of space-times. KY two and three forms are taken up in the next paper [20].

Although KY-forms have been the subject of relatively recent active research, Killing vectors have been so intensely investigated that the following original points which hold for the KY-forms as well are worth emphasizing. (i) As has been mentioned above, there exist five well-known space-times that are covered by the considered class of metrics such that one of them (de Sitter) consists of six different types. Killing vector fields of these metrics are usually handled as case-by-case studies in scattered references. Our unified

generation may remedy many inconveniences such as notational incompatibilities, proper range problems related to coordinates and relationships between cases. Moreover, one can explicitly observe the emergence of each case from the variations of the metric characterizing coefficients. (ii) Derivations of Killing vector fields for some types of de Sitter space-time from a five dimensional embedding flat manifold is geometrically very appealing and more inspiring physically (for a review of de Sitter spaces see [21, 22] and for recent interest see [23] and references therein). But the exact number of possible types naturally emerges from our study, and we identify an exactly solvable equation that determines this number by the number of its possible solutions. (iii) Our approach is exhaustive in the sense that all of the possible Killing vector fields of a given space-time can be completely determined from our approach so long as its metric belongs to the considered class. This fact makes it possible to reach decisive, or at least conclusive statements about a particular problem in which KY-forms are involved. We do not go into the detail of all possible cases but point out sufficiently symmetric cases, and have given some details of a particular case having six independent Killing vector fields that, as far as we know, does not appear in the literature. This example is also worth mentioning in light of the fact that its symmetry algebra changes drastically when a seemingly unimportant integration constant is changed.

II. KY 1-FORMS: DEFINING EQUATIONS

A 1-form is a KY 1-form if and only if it is the metric dual of a Killing vector field. This is equivalent to the fact that it satisfies equation (1) for $p = 1$. For the components of

$$\omega_{(1)} = \alpha e^0 + \beta e^1 + \gamma e^2 + \delta e^3 \ ,$$

the KY equation gives, in view of (2) and (3), sixteen equations of which the following ten

$$\begin{aligned}
\alpha_t &= \frac{H_0'}{TH_1}\beta \ , & \beta_r &= \dot{T}\frac{H_1}{H_0}\alpha \ , \\
T^2\partial_t\frac{\beta}{T} &= -\frac{H_0^2}{H_1}\partial_r\frac{\alpha}{H_0} \ , & \beta_\theta &= -\frac{H_2^2}{H_1}\partial_r\frac{\gamma}{H_2} \ , \\
T^2\partial_t\frac{\gamma}{T} &= -\frac{H_0}{H_2}\alpha_\theta \ , & \beta_\varphi &= -\frac{H_2^2}{H_1}\sin\theta\partial_r\frac{\delta}{H_2} \ , \\
T^2\partial_t\frac{\delta}{T} &= -\frac{H_0}{H_2\sin\theta}\alpha_\varphi \ , & \gamma_\theta &= \dot{T}\frac{H_2}{H_0}\alpha - \frac{H_2'}{H_1}\beta \ , \\
\delta_\varphi &= \sin^2\theta\partial_\theta\frac{\gamma}{\sin\theta} \ , & \gamma_\varphi &= -\sin^2\theta\partial_\theta\frac{\delta}{\sin\theta}
\end{aligned} \tag{4}$$

are independent. Although it appears difficult to solve this coupled set of first order partial differential equations directly, an exhaustive treatment of the problem with many classes of solutions is possible.

At first to gain an initial insight into the above set of equations, let us look at some obvious solutions. We can immediately see that when all coefficient functions but α are zero, T must be constant and $\alpha = H_0$. When β alone is nonzero and proportional to T, the conditions $H_0' = 0 = H_2'$ must be fulfilled. The other two cases, in which only γ or only δ is different from zero and proportional to TH_2, are forbidden by the last two equations of (4). But it is easy to verify that

$$\alpha = 0 = \beta \ , \quad \gamma = TH_2 g \ , \quad \delta = TH_2(a_3 \sin\theta + \cos\theta g_\varphi) \ ,$$

is a solution, such that $g(\varphi)$ satisfies $g_{\varphi\varphi} + g = 0$ and there is no additional constraint. We thus have, for $g = a_1 \sin\varphi - a_2 \cos\varphi$, the following three linearly independent KY 1-forms :

$$\begin{aligned} \tilde{K}_1 &= TH_2(\sin\varphi e^2 + \cos\theta\cos\varphi e^3) \ , \\ \tilde{K}_2 &= TH_2(-\cos\varphi e^2 + \cos\theta\sin\varphi e^3) \ , \\ \tilde{K}_3 &= -TH_2 \sin\theta e^3 \ . \end{aligned} \tag{5}$$

The corresponding rotational Killing vector fields

$$\begin{aligned} K_1 &= \sin\varphi\partial_\theta + \cot\theta\cos\varphi\partial_\varphi \ , \\ K_2 &= -\cos\varphi\partial_\theta + \cot\theta\sin\varphi\partial_\varphi \ , \\ K_3 &= -\partial_\varphi \ , \end{aligned} \tag{6}$$

are the well-known generators of the $so(3)$-algebra: $[K_i, K_j] = \varepsilon_{ij}{}^k K_k$. As there is no constraint on these solutions, they must appear in every case independent of the specific form of the functions characterizing the metric. This is a typical characteristic of spherical symmetry.

When $\dot{T} = 0$, $\omega_0 = H_0 e^0$ is a KY 1-form which corresponds to the time-like Killing vector field $K_0 = \tilde{\omega}_0 = -\partial_t$, and it can be combined with the above $so(3)$ solutions. In fact for $\alpha = H_0$, $\beta = 0$ and $\dot{T} = 0$, the above $4D$ algebra is the dual of the most general solution of equations (4). Indeed in such a case, the first eight equations of (4) imply that $\gamma = TH_2 G(\varphi)$ and $\delta = TH_2 D(\theta, \varphi)$ and then the last two equations of (4) yield

$$D_\varphi = -\cos\theta G \ , \quad G_\varphi = -\sin\theta D_\theta + \cos\theta D \ .$$

From the derivation of the second equation with respect to φ, we obtain $G_{\varphi\varphi} + G = 0$ in view of the first equation. On the other hand, integration of the first equation gives $D = \cos\theta G_\varphi + f(\theta)$ which, upon substituting it into the second, gives $f_\theta = \cot\theta f$ whose integral is $f = a_3 \sin\theta$. In the case of $H_2 = r$, such space-times with $4D$ symmetry algebra include two physically important examples: the Reissner-Nordström (RN) and its special case Schwarzschild space-times, for which the other coefficient functions are given in Table I. It should be emphasized that the explicit forms of H_0 and H_1 are derived from the physical requirements, namely from the Einstein equations in the Schwarzschild case.

For a general consideration, it turns out to be convenient to look for the solutions in the set of the solutions of some integrability conditions. This will also provide us with the necessary means to generate other sets of solutions.

III. INTEGRABILITY CONDITIONS AND THEIR SOLUTIONS

For $x = \alpha, \beta$ we have the integrability conditions

$$\partial_\varphi \partial_\theta (\frac{x}{\sin\theta}) = 0 \ , \quad x_{\varphi\varphi} = \sin^3\theta \partial_\theta \frac{x_\theta}{\sin\theta} \ , \tag{7}$$

that follow from $\delta_{r\theta} = \delta_{\theta r}, \delta_{rt} = \delta_{tr}, \delta_{t\theta} = \delta_{\theta t}, \delta_{t\varphi} = \delta_{\varphi t}$ and $\gamma_{r\varphi} = \gamma_{\varphi r}, \gamma_{\theta\varphi} = \gamma_{\varphi\theta}$. There are two additional conditions for α and four conditions for γ and δ:

$$\begin{aligned} \partial_r \partial_\theta (\frac{\alpha}{H_2}) \ &= \ 0 = \partial_r \partial_\varphi (\frac{\alpha}{H_2}) \ , \\ \partial_r \partial_\theta (\frac{\delta}{H_2}) \ &= \ 0 = \partial_t \partial_\theta (\frac{\delta}{T}) \ , \\ \partial_t \partial_r (\frac{\delta}{TH_0}) \ &= \ 0 = \partial_t \partial_r (\frac{\gamma}{TH_0}) \ . \end{aligned} \tag{8}$$

The first two conditions of (8) can be checked from $\alpha_{r\theta} = \alpha_{\theta r}$ and $\alpha_{r\varphi} = \alpha_{\varphi r}$. The second row of (8) follows from $\alpha_{\theta\varphi} = \alpha_{\varphi\theta}, \beta_{\theta\varphi} = \beta_{\varphi\theta}$ and the last row can be seen from $\beta_{t\varphi} = \beta_{\varphi t}$ and $\beta_{r\theta} = \beta_{\theta r}$. The following two equations can be easily verified from the last row of (4)

$$y_{\varphi\varphi} + y = -\sin^3\theta \partial_\theta \frac{y_\theta}{\sin\theta} \ , \tag{9}$$

for $y = \gamma, \delta$. There are 24 integrability conditions but only 20 of them are independent, and the twelve shown above are sufficient for a unified and exhaustive investigation.

1. The General Forms of α and β

In terms of the functions $f = f(t, r, \varphi), g = g(t, r, \theta)$ the first equation of (7) implies

$$x_\varphi = \sin\theta f \ , \quad x_\theta = \cot\theta x + g \ ,$$

and from the second equation of (7) we obtain $x = \sin^2\theta g_\theta - \sin\theta f_\varphi$. On substituting this solution into the above x_φ and x_θ equations we arrive at

$$f_{\varphi\varphi} + f = 0 \ , \ g = \sin\theta\partial_\theta(\sin\theta g_\theta) \ ,$$

whose general solutions can be written, with $f_i = f_i(t, r)$ and $g_i = g_i(t, r)$, as

$$f = -f_1 \sin\varphi - f_2 \cos\varphi \ , \quad g = -g_1 \cot\theta - g_2 \frac{1}{\sin\theta} \ .$$

Here the minus signs are used for convenience. The general solution for x is

$$x = g_1 + g_2 \cos\theta + \sin\theta(f_1 \cos\varphi - f_2 \sin\varphi) \ .$$

In terms of the functions $A_i = A_i(t, r)$ and $B_i = B_i(t, r)$ let us define

$$\begin{aligned} \sigma^A &= A_1 \cos\varphi - A_2 \sin\varphi \ , \quad \sigma^B = B_1 \cos\varphi - B_2 \sin\varphi \ , \\ A &= \sin\theta\sigma^A + A_3 \cos\theta \ , \qquad B = \sin\theta\sigma^B + B_3 \cos\theta \ . \end{aligned}$$

Since the functions f_i and g_i are, in general, different for α and β, we can write their general forms very concisely as

$$\alpha = U + A \ , \quad \beta = V + B \ , \tag{10}$$

where U and V depend, like the g_1 term of x, on t and r. Note that A and B depend on all of the coordinates and they satisfy the relations

$$A_{\theta\theta} + A = 0 = B_{\theta\theta} + B \ , \quad \sigma^A_{\varphi\varphi} + \sigma^A = 0 = \sigma^B_{\varphi\varphi} + \sigma^B \ . \tag{11}$$

In the case of $x = \alpha$, the first two conditions of (8) imply that the functions characterizing A are proportional to H_2 and these enable us to write $A = H_2\xi$ such that

$$\xi = u \cos\theta + \sin\theta(v_1 \cos\varphi - v_2 \sin\varphi) \ , \tag{12}$$

where u, v_i depend only on t.

2. The General Forms of γ and δ

When α and β given by (10) are substituted into the γ_θ-equation of (4), we obtain

$$\gamma_\theta = (\dot{T}\frac{H_2}{H_0}U - \frac{H_2'}{H_1}V) + \dot{T}\frac{H_2}{H_0}A - \frac{H_2'}{H_1}B \ .$$

For well-defined γ solutions, the first term at the right hand side must be zero:

$$\dot{T}U = H_0 PV \ , \quad (P = \frac{H_2'}{H_1 H_2}) \ . \tag{13}$$

(Since the mentioned term is independent of θ it would lead, upon integration, to a γ solution which linearly depends on θ. In fact this condition results when the above γ_θ is used in eq. (9) of γ.) Then, in view of eq.(11), the γ_θ-equation can be integrated to

$$\gamma = -(\dot{T}\frac{H_2}{H_0}A_\theta - \frac{H_2'}{H_1}B_\theta) + TH_2 g \ , \ \ g_{\varphi\varphi} + g = 0 \ , \tag{14}$$

where the θ-independent term $G = TH_2 g(\varphi)$ comes from integration, and its form can be easily verified by substituting γ in the α_θ and β_θ-equations of (4). Indeed, in the first case we get

$$T^2 \partial_t (\frac{\dot{T}}{T}A_\theta) - \frac{H_0^2}{H_2^2}A_\theta = T^2 H_0 P \partial_t \frac{B_\theta}{T} \ , \tag{15}$$

in addition to $\partial_t(G/T) = 0$. In the second case, we obtain

$$\dot{T}\partial_r(\frac{A_\theta}{H_0}) = \frac{H_1}{H_2^2}B_\theta + \partial_r(PB_\theta) \ , \tag{16}$$

and $\partial_r(G/H_2) = 0$. These imply that G is of the form $G = TH_2 g(\varphi)$ and by the integrability condition (9) for γ, we see that g must satisfy the equation given by (14).

Having determined the general form of γ, we now use it in the last two equations of (4) to determine the form of δ. The δ_φ-equation of (4) directly gives

$$\delta_\varphi = \dot{T}\frac{H_2}{H_0}\sigma^A - \frac{H_2'}{H_1}\sigma^B - TH_2 \cos\theta g \ ,$$

which can be integrated to

$$\delta = -\dot{T}\frac{H_2}{H_0}\sigma^A_\varphi + \frac{H_2'}{H_1}\sigma^B_\varphi + TH_2\cos\theta g_\varphi + aTH_2 \sin\theta \ , \tag{17}$$

where a is an integration constant and the φ-independent term $D = aTH_2 \sin\theta$ again comes from integration, whose explicit form is determined from α_φ, β_φ and the last two equations

of (4). Indeed, the α_φ-equation of (4) states that D/T must be independent of t and that the following condition must be satisfied :

$$T^2\partial_t(\frac{\dot{T}}{T}\sigma_\varphi^A) - \frac{H_0^2}{H_2^2}\sigma_\varphi^A = T^2 H_0 P \partial_t \frac{\sigma_\varphi^B}{T} \ . \tag{18}$$

On the other hand, the β_φ-equation of (4) states that D/H_2 must be independent of r, and that the following condition must be satisfied :

$$\dot{T}\partial_r \frac{\sigma_\varphi^A}{H_0} = \frac{H_1}{H_2^2}\sigma_\varphi^B + \partial_r(P\sigma_\varphi^B) \ . \tag{19}$$

As a particular result, we have $D = TH_2 f(\theta)$ and when the found forms of γ and δ are used in the last equation of (4), we obtain $f = a\sin\theta$. Note that the conditions (18) and (19) are contained in (15) and (16).

As an intermediate result, all the metric coefficient functions have been specified in terms of seven functions U, A_i and B_i, which can be determined from the first three equations of (4) and the conditions found in (13), (15) and (16). Therefore, seven equations of (4) have been analyzed, and the first three equations and conditions (15) and (16) remain to be solved. Note that the last terms of γ and δ correspond to the $so(3)$ solutions. The rest of the investigation is entirely devoted to the specification of additional symmetries.

3. *The Clustering of Solutions*

We shall now substitute the solutions (10) and (12) into the first three equations of (4) to specify the unknown functions. The first three equations of (4) give the following three equations for U

$$\begin{aligned} U_t &= \frac{\dot{T}}{T}\frac{H_0' H_2}{H_0 H_2'}U \ , \\ \partial_r(\frac{U}{H_0 P}) &= H_1 \frac{U}{H_0} \ , \\ T^2\partial_t(\frac{\dot{T}}{T}U) &= -\frac{H_0^3 P}{H_1}\partial_r\frac{U}{H_0} \ , \end{aligned} \tag{20}$$

and the following three equations for B

$$B = TL\xi_t \ , \quad B_r = \dot{T}\frac{H_1 H_2}{H_0}\xi \ , \quad T^2\partial_t\frac{B}{T} = -\frac{H_0^2}{H_1}(\frac{H_2}{H_0})'\xi \ , \tag{21}$$

where we have utilized relation (13) and the function $L = L(r)$ is defined by

$$L = \frac{H_1 H_2}{H_0'} \ . \tag{22}$$

As is obvious from the equations in this subsection, from here on the analysis critically depends on the derivatives of T, H_0 and H_2. Since H_2' is different from zero in all physically important space-times, we shall assume this to be the case throughout the paper. This means that the function P is different from zero. The cases $H_0' = 0$ and $\dot{T} = 0$ will be considered as particular cases in the last three sections. In the next section, we proceed to look for solutions for which both H_0' and $\dot{T}$ can be different from zero.

IV. THREE CLASSES OF SOLUTIONS

Using the first equation of (21) in the other two equations of (21), we obtain two equations which accept separation of variables such that

$$m\xi_t = \frac{\dot{T}}{T}\xi \ , \quad T^2 \xi_{tt} = -m_0 \xi \ , \tag{23}$$

provided that m and m_0 defined by

$$m = L' \frac{H_0}{H_1 H_2} \ , \quad m_0 = \frac{H_0^2}{L H_1} (\frac{H_2}{H_0})' \ , \tag{24}$$

are constants. When one side of a separable equation depends entirely on the metric coefficient functions, we shall use the letters k, l and m for separation constants. Other integration constants will be denoted by the letters a, b and c.

In view of the ansatz $U = H_0 Y(r) f(t)$ the first two equations of (20) transform to

$$f_t = k\frac{\dot{T}}{T} f \ , \quad (\frac{Y}{P})' = H_1 Y \ , \tag{25}$$

where the metric constant k is defined by

$$k = \frac{H_0' H_2}{H_0 H_2'} \ . \tag{26}$$

On the other hand from the last equation of (20), with the same ansatz, we get

$$T^2 \partial_t (\frac{\dot{T}}{T} f) = k_0 f \ , \quad \frac{Y'}{Y} = -k_0 \frac{H_1}{H_0^2 P} \ , \tag{27}$$

where k_0 is a separation constant. Two equations of (25) can be easily integrated to

$$f = c_1 T^k \; , \quad Y = c_2 H_2 P \; . \tag{28}$$

Thus $U = cH_0H_2PT^k$ where c_i are integration constants and $c = c_1c_2$. When these solutions are substituted into (27) we see that T must satisfy

$$T\ddot{T} + (k-1)\dot{T}^2 = k_0 \; , \tag{29}$$

and k_0 must be the metric constant

$$k_0 = -\frac{H_0^2}{H_1H_2}(\frac{H_2'}{H_1})' \; . \tag{30}$$

Noting that equations (20) and (21) are linear in U and B, depending on the values of m one can distinguish three classes of solutions for which both H_0' and $\dot{T}$ can be different from zero. These can be characterized as follows

$$(A)\;\; m = 0 \; , \quad (B)\;\; m \neq 0 \; ; \;\; U = 0 = V \; , \quad (C)\;\; m \neq 0 \; .$$

The cases (i) $\dot{T} = 0, \; H_0' \neq 0$, (ii) $\dot{T} \neq 0, \; H_0' = 0$ and (iii) $\dot{T} = 0 = H_0'$ will be considered separately in the last three sections.

A. $m = 0$ Solutions

When m is zero L is a nonzero constant and we have $H_0'L = H_1H_2$. In that case, equations given by (23) imply that $\xi = 0 = B$ and that m_0 need not be a constant. The conditions (15) and (16) are both trivially satisfied for this case and hence, γ and δ solutions correspond only to the $so(3)$ solutions. Only the metric constants k and k_0 are defined in that case and, provided that T satisfies equation (29), we have the following solutions for α and β:

$$\alpha = U = cT^kH_0\frac{H_2'}{H_1} \; , \quad \beta = V = c\dot{T}T^kH_2 \; , \tag{31}$$

which determine the Killing vector field

$$X = -\frac{H_2'}{H_1}T^k\partial_t + \dot{T}T^{k-1}\frac{H_2}{H_1}\partial_r \; , \tag{32}$$

commuting with the $so(3)$ vector fields.

B. $m \neq 0$ and $U = 0 = V$ Solutions

When m is different from zero, the first equation of (23) can be easily solved to be

$$\xi = T^{1/m}\zeta \ , \quad \zeta = a_1 \cos\theta + \sin\theta(a_2 \cos\varphi - a_3 \sin\varphi) \ , \tag{33}$$

where a_i are integration constants. The second equation of (23) then implies that

$$T^{(2m-1)/m}\partial_t(\dot{T}T^{(1-m)/m}) = -mm_0 \ . \tag{34}$$

This equation can equivalently be read as

$$(1-m)\dot{T}^2 + mT\ddot{T} = -m^2 m_0 \ , \tag{35}$$

or more concisely, in terms of $K = T^{1/m}$, as

$$\ddot{K} = -m_0 K^{1-2m} \ . \tag{36}$$

The corresponding α, β, γ and δ solutions are as follows:

$$\begin{aligned} \alpha &= T^{1/m}H_2\zeta \ , \quad \beta = WL\zeta \ , \\ \gamma &= \frac{1}{m_0}WH_0\zeta_\theta + TH_2 g \ , \quad (g_{\varphi\varphi} + g = 0) \\ \delta &= \frac{1}{m_0}WH_0\sigma_\varphi + TH_2(a\sin\theta + \cos\theta g_\varphi) \ , \end{aligned} \tag{37}$$

where $W = \dot{T}T^{1/m}/m = \dot{K}T$ and

$$\sigma^A = T^{1/m}H_2\sigma \ , \quad \sigma^B = WL\sigma \ , \quad \sigma = a_2\cos\varphi - a_3\sin\varphi \ .$$

In view of (24) and equation (35); it is easy to verify that while condition (15) yields,

$$m_0\left(\frac{H_2'}{H_0'} - m\frac{H_2}{H_0}\right) = \frac{H_0}{H_2} \ , \tag{38}$$

for $A = T^{1/m}H_2\zeta$ and $B = WL\zeta$, condition (16) yields an equation that is just the derivative of (38). In writing γ and δ of (37) the condition (38) has been used.

C. $m \neq 0$ Solutions

In this case, solutions are just a combination of the above two classes: γ and δ solutions are as in equations (37) but to the α and β solutions of (37), one must add that given by

(31). Equation (38) also hold in this case. However, there are four metric constants m, m_0, k and k_0, and T must satisfy both the equations (29) and (34) (or equivalently (35) or (36)). A detailed investigation of the four metric constants presented in Appendix A shows that for H_2' to be nonzero, the following conditions must be satisfied:

$$km = 1 \ , \quad m = 1 + m_0\ell^2 \ ,$$

where ℓ is another metric constant. In fact, from equations (29), (35) and the above relation, we also get $k_0 = -mm_0$. To all these one must also add relation (38).

In the case of (B) class solutions, we have six arbitrary real constants: a and two constants determined by the function g and three constants given by ζ. These mean that we have six linearly independent Killing vector fields for class (B) which will be presented together with their Lie algebra in the next subsection. In the case of (C), which includes (A) and (B) solutions as special subcases, we have only one additional symmetry generator given by (32). Although some other subcases of (C) can be defined, this case will not be pursued any further as it has many additional conditions.

D. Killing Vector Fields and Lie Algebra for the case (B)

In terms of the two nonzero constants m and m_0 defined by (24), we have obtained, in addition to three $so(3)$ 1-forms given by (5), three additional linearly independent KY 1-forms:

$$\begin{aligned}
\omega_1 &= \cos\theta\omega - \frac{1}{m_0}WH_0\sin\theta e^2 \ , \\
\omega_2 &= \sin\theta\cos\varphi\omega + \frac{1}{m_0}WH_0(\cos\theta\cos\varphi e^2 - \sin\varphi e^3) \ , \\
\omega_3 &= -\sin\theta\sin\varphi\omega - \frac{1}{m_0}WH_0(\cos\theta\sin\varphi e^2 + \cos\varphi e^3) \ ,
\end{aligned} \tag{39}$$

where the 1-form ω is defined by

$$\omega = T^{1/m}H_2e^0 + WLe^1 \ . \tag{40}$$

By noting that the metric dual of e^0 is $-H_0^{-1}\partial_t$, in terms of $K = T^{1/m}$ the vector field X which is the metric dual of ω can be written as

$$X = \tilde{\omega} = -K\frac{H_2}{H_0}\partial_t + \dot{K}\frac{H_2}{H_0'}\partial_r \ . \tag{41}$$

The Killing vector fields corresponding to (39) are then given by

$$\begin{aligned} X_1 &= \cos\theta X - MZ_1 \ , \\ X_2 &= \sin\theta\cos\varphi X + MZ_2 \ , \\ X_3 &= -\sin\theta\sin\varphi X - MZ_3 \ , \end{aligned} \tag{42}$$

where $M = \dot{K}H_0/m_0H_2$ and the vector fields Z_i are defined as

$$Z_1 = \sin\theta\partial_\theta \ , \quad Z_2 = \cos\theta\cos\varphi\partial_\theta - \frac{\sin\varphi}{\sin\theta}\partial_\varphi \ , \quad Z_3 = \cos\theta\sin\varphi\partial_\theta + \frac{\cos\varphi}{\sin\theta}\partial_\varphi \ . \tag{43}$$

By making use of

$$[Z_1, Z_2] = K_2 \ , \qquad [Z_1, Z_3] = -K_1 \ , \quad [Z_2, Z_3] = -K_3 \ , \tag{44}$$

$$\begin{aligned} [Z_1, K_1] &= -Z_3 \ , \quad [Z_1, K_2] = Z_2 \ , \qquad [Z_1, K_3] = 0 \ , \\ [Z_2, K_1] &= 0 \ , \qquad [Z_2, K_2] = -Z_1 \ , \quad [Z_2, K_3] = -Z_3 \ , \\ [Z_3, K_1] &= Z_1 \ , \quad [Z_3, K_2] = 0 \ , \qquad [Z_3, K_3] = Z_2 \ , \end{aligned} \tag{45}$$

we obtain

$$\begin{aligned} [K_1, X_1] &= X_3 \ , \qquad [K_1, X_2] = 0 \ , \qquad [K_1, X_3] = -X_1 \ , \\ [K_2, X_1] &= X_2 \ , \qquad [K_2, X_2] = -X_1 \ , \qquad [K_2, X_3] = 0 \ , \\ [K_3, X_1] &= 0 \ , \qquad [K_3, X_2] = -X_3 \ , \qquad [K_3, X_3] = X_2 \ . \end{aligned} \tag{46}$$

On the other hand in terms of $s = X(M) + M^2$ we have

$$[X_1, X_2] = sK_2 \ , \quad [X_1, X_3] = -sK_1 \ , \quad [X_2, X_3] = -sK_3 \ . \tag{47}$$

For evaluation of s, we should recall that K obeys the equation (36). For $m = 1$ we have $K = T$ and $T\ddot{T} = -m_0$, which can be integrated to $\dot{T}^2 + 2m_0 \ln T = \ell_0$. For $m \neq 1$, it can also be easily integrated once to obtain

$$\dot{K}^2 = -\frac{m_0}{1-m}K^{2(1-m)} + m_2 \ , \tag{48}$$

where ℓ_0 and m_2 are integration constants determined by the metric. We can therefore write

$$\begin{aligned} s &= -K\frac{H_2}{H_0}M_t + \dot{K}\frac{H_2}{H_0'}M_r + M^2 \ , \\ &= \frac{1}{m_0}[-K\ddot{K} + (1-m)\dot{K}^2] \ , \\ &= \begin{cases} \frac{1-m}{m_0}m_2 \ , & \text{for} \quad m \neq 1, \\ 1 \ , & \text{for} \quad m = 1 \ , \end{cases} \end{aligned} \tag{49}$$

where we have made use of (38) in the second line. When $m \neq 1$ and $s \neq 0$ the X_i generator can be normalized with the same constant such that at the right hand side of (47), there appear $\pm K_j$. Such a normalization does not affect relations (46). But when the integration constant m_2 is zero, the generator set $\{X_1, X_2, X_3\}$ form an abelian subalgebra. In that case the symmetry algebra is isomorphic to $3D$ Euclidean algebra $e(3)$. This shows how the symmetry of the metric may change for different values of an integration constant.

V. MAXIMAL SYMMETRIES : $\dot{T} = 0,\ H_0' \neq 0$ SOLUTIONS

Let us begin this case by considering α and β as given in equation (10), such that $A = (H_0'/H_1)\xi$ and the functions $V,\ V_i,\ u$ and v_i depend only on $\tau = t/T$ for β is independent of r. Since $\dot{T} = 0$ in this case, condition (13) requires that $V = 0$ and we can then write, from the first equation of (4):

$$\alpha = U + \frac{H_0'}{H_1}\xi\ , \quad \beta = \xi_\tau\ , \tag{50}$$

where ξ is given by (12) and $U_\tau = 0$. The third equation of (4) now provides us with

$$U = c_0 H_0\ , \quad \xi_{\tau\tau} + k_1\xi = 0\ , \tag{51}$$

where c_0 is an integration constant, and the metric constant k_1 is defined by

$$k_1 = \frac{H_0^2}{H_1}(\frac{H_0'}{H_0 H_1})'\ . \tag{52}$$

Note that the second equation of (51) is equivalent to three similar equations for u, v_1 and v_2, each of which gives two linearly independent solutions depending on the value of k_1.

Having completely specified α and β with $A = H_0'\xi/H_1$ and $B = \xi_\tau$, we now turn to conditions (15) and (16), which in this case amount to

$$H_0 H_0' = k_1 H_2 H_2'\ , \quad P' = -\frac{H_1}{H_2^2}. \tag{53}$$

k_1 is a nonzero metric constant for $H_0' \neq 0$. Both of these relations can be integrated to

$$H_0^2 = k_1 H_2^2 + k_2\ , \quad P^2 = k_3 + \frac{1}{H_2^2}\ , \tag{54}$$

where k_2 and k_3 are integration constants. The first relation of (54) implies that for $H_0' \neq 0$, k_1 must be different from zero. The conditions (52) and (53) also imply that

$$P^2 = \frac{H_0^2}{k_2 H_2^2}\ , \quad k_1 = k_2 k_3\ , \tag{55}$$

which means for $H_2' \neq 0,\ k_2$ and k_3 must be different from zero as well.

If k_1 is a negative constant such that $k_1 = -\kappa^2$ where κ is a nonzero real number, we can write, in terms of integration constants c_i, the ξ solutions of (51) as

$$\begin{aligned}
\xi &= (c_1 \cosh \kappa\tau + c_2 \sinh \kappa\tau) \cos\theta + \sin\theta \sigma^A \ , \\
\sigma^A &= (c_3 \cosh \kappa\tau + c_4 \sinh \kappa\tau) \cos\varphi - (c_5 \cosh \kappa\tau + c_6 \sinh \kappa\tau) \sin\varphi \ .
\end{aligned}$$

Then γ and δ can be determined from (14) and (17) as

$$\gamma = \frac{H_2'}{H_1}\xi_{\tau\theta} + TH_2 g \ , \quad \delta = \frac{H_2'}{H_1}\sigma^A_{\tau\varphi} + TH_2(a\sin\theta + \cos\theta g_\varphi) \ . \tag{56}$$

The solutions (50) and (56) determine seven linearly independent KY 1-forms in addition to $so(3)$ solutions. The first one is $\omega_0 = H_0 e^0$ which corresponds to $K_0 = -\partial/\partial t$. For $k_1 = -\kappa^2$, we can write these additional forms as follows:

$$\begin{aligned}
\omega_1 &= \cos\theta\psi_1 - \kappa\frac{H_2'}{H_1}\sinh\kappa\tau\sin\theta e^2 \ , \qquad \omega_2 = \cos\theta\psi_2 - \kappa\frac{H_2'}{H_1}\cosh\kappa\tau\sin\theta e^2 \ , \\
\omega_3 &= \sin\theta\cos\varphi\psi_1 + \kappa\frac{H_2'}{H_1}\sinh\kappa\tau\phi_1 \ , \qquad \omega_4 = \sin\theta\cos\varphi\psi_2 + \kappa\frac{H_2'}{H_1}\cosh\kappa\tau\phi_1 \ , \\
\omega_5 &= -\sin\theta\sin\varphi\psi_1 - \kappa\frac{H_2'}{H_1}\sinh\kappa\tau\phi_2 \ , \quad \omega_6 = -\sin\theta\sin\varphi\psi_2 - \kappa\frac{H_2'}{H_1}\cosh\kappa\tau\phi_2 \ ,
\end{aligned} \tag{57}$$

where the 1-forms ψ_i and $\phi_i,\ i = 1, 2$ are defined by

$$\begin{aligned}
\psi_1 &= \cosh\kappa\tau\frac{H_0'}{H_1}e^0 + \kappa\sinh\kappa\tau e^1 \ , \quad \phi_1 = \cos\theta\cos\varphi e^2 - \sin\varphi e^3 \ , \\
\psi_2 &= \sinh\kappa\tau\frac{H_0'}{H_1}e^0 + \kappa\cosh\kappa\tau e^1 \ , \quad \phi_2 = \cos\theta\sin\varphi e^2 + \cos\varphi e^3 \ .
\end{aligned}$$

The vector fields $W_i = \tilde{\psi}_i$ and $\bar{Z}_{i+1} = \tilde{\phi}_i$ are, in terms of Z_2 and Z_3 given by (43), as follows:

$$W_1 = -\cosh\kappa\tau\frac{H_0'}{TH_0H_1}\partial_\tau + \frac{\kappa}{TH_1}\sinh\kappa\tau\partial_r \ , \quad \bar{Z}_2 = \frac{1}{TH_2}Z_2 \ , \tag{58}$$

$$W_2 = -\sinh\kappa\tau\frac{H_0'}{TH_0H_1}\partial_\tau + \frac{\kappa}{TH_1}\cosh\kappa\tau\partial_r \ , \quad \bar{Z}_3 = \frac{1}{TH_2}Z_3 \ . \tag{59}$$

It is also not difficult to verify that for $H_2 = r$ we have, from (54)

$$H_0^2 = k_1 r^2 + k_2 \ , \quad H_1^2 = \frac{1}{1 + k_3 r^2} \ ,$$

and $H_0H_1 = 1$ for $k_2 = 1$. In the case of $k_1 = -1 = k_3$ and $k_2 = 1$, we recover the static form of the de Sitter metric (see [2] pp.492). KY 1-forms for five different forms of the de Sitter type space-times are obtained in the next section.

VI. de SITTER AND ROBERTSON-WALKER TYPE SYMMETRIES: $H_0' = 0, \dot{T} \neq 0$

Since $\alpha_t = 0$ in this case, it is convenient to start by defining the constant

$$\ell = T^2 \partial_t \frac{\dot{T}}{T} \; . \tag{60}$$

The nonzero and zero values of ℓ will then be considered separately. In these two cases, T is restricted to be a special function of time by the symmetry requirement.

Case A considered below leads us to a family of de Sitter type space-times with ten independent Killing vector fields and, depending on the values of ℓ and other integration constants, four different time evolution regimes can explicitly be specified by the symmetry requirement. The B case, specified by $\ell = 0$, corresponds to the best known form of de Sitter space-time, again having ten independent Killing vector fields such that T is an exponential function of time. However, there is an important special case specified by $\alpha = 0$, and therefore T is not restricted by any symmetry requirement. This corresponds to the Robertson-Walker space-time with six dimensional symmetry algebra in which they are the Einstein equations that give the time dependence as shown in Table I.

A. The case $\ell \neq 0$

We start with

$$\alpha = U(r) + H_2 \zeta \; , \quad \beta = \dot{T} Y(r) + B \; , \tag{61}$$

where ζ is given by (33) and B is defined as in Section III. Condition (13) implies that $U = H_0 P Y$ and the second and third equations of (4) then yield

$$Y' \;=\; H_1 P Y \; , \qquad\qquad Y = -\frac{H_0^2}{\ell H_1}(PY)' \; , \tag{62}$$

$$B_r \;=\; \dot{T} \frac{H_1 H_2}{H_0} \zeta \; , \;\; T^2 \partial_t \frac{B}{T} = -H_0 H_2 P \zeta \; . \tag{63}$$

These are the reduced forms of equations (20) and (21). The first equation of (62) gives $Y = c_1 H_2$ and from the second equation we then obtain

$$P' = -H_1 (P^2 + \frac{\ell}{H_0^2}) \; . \tag{64}$$

The two equations of (63) imply that, in terms of

$$\eta_2 = b_1 \cos\theta + \sin\theta\sigma^b \ , \quad \sigma^b = b_2 \cos\varphi - b_3 \sin\varphi \ , \tag{65}$$

the most general solution for B is of the form $B = \dot{T}\eta_1(r,\theta,\varphi) + T\eta_2(\theta,\varphi)$. The second equation of (63) specifies η_1 in terms of ζ:

$$\eta_1 = -\frac{H_0}{\ell} H_2 P\zeta \ , \tag{66}$$

and the first equation yields nothing but condition (64).

Having completely specified α and β with $A = H_2\zeta$ and

$$B = -\frac{H_0}{\ell}\dot{T}H_2P\zeta + T\eta_2 = \sin\theta\sigma^B + \cos\theta(b_1 T - \frac{H_0}{\ell}\dot{T}H_2P\sigma) \ , \tag{67}$$

such that

$$\sigma^B = T\sigma^b - \frac{1}{\ell}\dot{T}H_0H_2P\sigma \ , \tag{68}$$

the condition (14) amounts to

$$P^2 = -\frac{\ell}{H_0^2} + \frac{1}{H_2^2} \ . \tag{69}$$

It is not difficult to verify that condition (64) is implied by (69), which also yields $P' = -H_1/H_2^2$. (Note that the integration of (64) results in an additional constant multiplying the second term of (69)). In view of this relation, condition (16) is identically satisfied. That is, we only have condition (69).

We can now turn to (14) and (17) to evaluate γ and δ, and the solutions can be collated as follows:

$$\begin{aligned}
\alpha &= c_1 H_0 H_2 P + H_2\zeta \ , \\
\beta &= c_1 \dot{T} H_2 + \sin\theta\sigma^B + \cos\theta(b_1 T - \frac{H_0}{\ell}\dot{T}H_2P\sigma) \ , \\
\gamma &= -\dot{T}\frac{H_2^2}{H_0}\zeta_\theta + \dot{T}H_2P[\cos\theta\sigma^B - \sin\theta(b_1 T - \frac{H_0}{\ell}\dot{T}H_2P\sigma)] + TH_2 g \ , \\
\delta &= -\dot{T}\frac{H_2}{H_0}\sigma_\varphi + H_2P\sigma^B_\varphi + TH_2(a\sin\theta + \cos\theta g_\varphi) \ .
\end{aligned} \tag{70}$$

Note that for $H_0 = 1$ and $H_2 = r$, condition (69) yields

$$H_1^2 = \frac{1}{1 - \ell r^2} \ . \tag{71}$$

These are the characteristics of the de Sitter and Robertson-Walker space-times. It should be emphasized that the right hand side of condition (69) must be positive, which reflects the fact that $1/r^2 > \ell$ is required to avoid any singularity in the corresponding space-times.

We now turn to the explicit evolution of T. By multiplying both sides of equation (60) by $\dot{T}/T^3$, it can be integrated to

$$\dot{T} = \epsilon(\ell_3 T^2 - \ell)^{1/2} , \tag{72}$$

and then by integrating once more we get

$$T = \begin{cases} (\frac{\ell}{\ell_3})^{1/2}\cosh(\epsilon\ell_3^{1/2}t + a) ; & \text{for } \ell_3 > 0 \ , \ \ell > 0 \ , \\ \frac{\ell_0}{\ell_3^{1/2}}\sinh(\epsilon\ell_3^{1/2}t + a) ; & \text{for } \ell_3 > 0 \ , \ \ell = -\ell_0^2 < 0 \ , \\ \frac{\ell_0}{k_0}\sin(\epsilon k_0 t + a) ; & \text{for } \ell_3 = -k_0^2 < 0 \ , \ \ell = -\ell_0^2 < 0 \ , \\ \epsilon\ell_0 t + a ; & \text{for } \ell_3 = 0, \ \ell = -\ell_0^2 < 0 \ , \end{cases} \tag{73}$$

where ℓ_3 and a are integration constants and $\epsilon = \pm 1$. Note that the third solution can be inferred from the second one and that all the corresponding KY 1-forms can explicitly be read from (70). All of these de Sitter space-times of which the fourth one is a flat space-time, are spaces of constant curvature with the curvature scalar given by ℓ_3 (see Appendix B).

B. de Sitter type Symmetries: $\ell = 0 = H_0'$

The condition $\ell = 0$ is equivalent to $\dot{T} = \lambda T$, that is to

$$T = T_0 \exp(\lambda t) ,$$

where λ is a metric constant and T_0 is an integration constant. In this case, to avoid excessive repetitions, we shall be content to present the solutions:

$$\begin{aligned} \alpha &= c_0 H_0 + \frac{1}{H_0 P}\zeta , \\ \beta &= \dot{T}[\frac{c_0}{P} + \frac{1}{2}(\frac{1}{\dot{T}^2} + \frac{1}{P^2})\zeta + b_1\cos\theta + \sin\theta\sigma^b] , \\ \gamma &= \dot{T}H_2P\{[-\frac{1}{H_0^2P^2} + \frac{1}{2}(\frac{1}{\dot{T}^2} + \frac{1}{P^2})]\zeta_\theta - b_1\sin\theta + \cos\theta\sigma^b\} + TH_2 g , \\ \delta &= \dot{T}H_2P\{[-\frac{1}{H_0^2P^2} + \frac{1}{2}(\frac{1}{\dot{T}^2} + \frac{1}{P^2})]\sigma_\varphi + \sigma^b_\varphi\} + TH_2(a\sin\theta + \cos\theta g_\varphi) , \end{aligned} \tag{74}$$

which provide us with ten-dimensional symmetry algebra. Here, σ^b is given by (65). The above solutions can be easily verified provided that the condition $H_2' = \epsilon H_1$ holds, which implies that $H_2 P = \epsilon$ and hence, $P' = -H_1 P^2$.

C. The Case $\alpha = 0$: Robertson-Walker Symmetries

When $\alpha = 0$, the second and third equations of (4) imply that $\beta_r = 0$ and

$$\beta = B = T(\sin\theta\sigma + a_3\cos\theta) \ , \tag{75}$$

with $\sigma = \sigma^B/T = a_1\cos\varphi - a_2\sin\varphi$. In that case, eq. (15) is identically satisfied and eq. (16) gives $P' = -H_1/H_2^2$, which can be integrated to yield the second relation of eq. (54) and provide us with

$$H_1^2 = \frac{1}{1 + k_3 r^2} \ , \tag{76}$$

for $H_0 = 1$ and $H_2 = r$. Then the solutions can be easily read from (14) and (17) to be

$$\begin{aligned}
\gamma &= T\frac{H_2'}{H_1}(\cos\theta\sigma - a_3\sin\theta) + TH_2 g \ , \\
\delta &= T\frac{H_2'}{H_1}\sigma_\varphi + TH_2(a\sin\theta + \cos\theta g_\varphi) \ .
\end{aligned}$$

With $\alpha = 0$ and β as in (75), the following Killing vector fields are obtained :

$$\begin{aligned}
I_1 &= \frac{1}{H_1}\sin\theta\cos\varphi\partial_r + PZ_2 \ , \\
I_2 &= -\frac{1}{H_1}\sin\theta\sin\varphi\partial_r - PZ_3 \ , \\
I_3 &= \frac{1}{H_1}\cos\theta\partial_r - PZ_1 \ .
\end{aligned} \tag{77}$$

These are the generalized translation generators which, in the usual cartesian coordinates read, respectively, as $H_1^{-1}\partial_x$, $-H_1^{-1}\partial_y$ and $H_1^{-1}\partial_z$. Together with the $so(3)$ generators, they close into the following Lie algebra structure:

$$\begin{aligned}
[I_1, I_2] &= k_3 K_3 \ , \qquad [I_2, I_3] = k_3 K_1 \ , \quad [I_1, I_3] = -k_3 K_2 \ , \\
[K_1, I_2] &= -I_3 = [K_2, I_1] \ , \quad [K_1, I_3] = I_2 = -[K_3, I_1] \ , \\
[K_2, I_3] &= I_1 = [K_3, I_2] \ , \qquad [K_i, I_i] = 0 \ , \quad i = 1, 2, 3 \ .
\end{aligned} \tag{78}$$

VII. FLAT SPACE-TIME SOLUTIONS: $\dot{T} = 0$, $H_0' = 0$

In this case, α is independent of $\tau = t/T$, β is independent of r, and condition (13) requires that $V = 0$. Therefore it is convenient to start with $\alpha = c_0 H_0 + \eta_1$ and $\beta = \eta_2$,

where c_0 is a constant and η_i are as in (65) of the previous section, such that η_1 is independent from τ and η_2 is independent from r. The third equation of (4) gives $\eta_{2\tau} = -H_0\eta_{1r}/H_1$. Since the left hand side of this equality depends on τ and is independent of r, but the right hand side depends on r and is independent of τ, both sides must be equal

$$\eta_{2\tau} = \zeta = -\frac{H_0}{H_1}\eta_{1r} \ , \tag{79}$$

where ζ is given by (33). From the first equality, we obtain $\eta_2 = \tau\zeta + \zeta^{(b)}$ where $\zeta^{(b)}$ is the same as ζ with $a = (a_1, a_2, a_3)$ replaced by $b = (b_1, b_2, b_3)$. On the other hand, for $A = \eta_1$ and $B = \tau\zeta + \zeta^{(b)}$, conditions (15) and (16) respectively yield

$$\eta_{1\theta} = -\frac{H_2^2 P}{H_0}\zeta_\theta \ , \quad P' = -\frac{H_1}{H_2^2} \ . \tag{80}$$

In view of the second equality of (79), we obtain $(H_2^2 P)' = H_1$ from the integrability condition $\eta_{1\theta r} = \eta_{1r\theta}$. From the second equation of (79) and $(H_2^2 P)' = H_1$, we get $P^2 = 1/H_2^2$, which can also be obtained from the integration of $P' = -H_1/H_2^2$ with the zero integration constant (see the discussion following equations (53) and (54)).

From (14) and (17), one can easily write out γ and δ together of α and β to be as :

$$\begin{aligned}
\alpha &= c_0 H_0 - \frac{1}{H_0 P}\zeta \ , \quad \beta = \tau\zeta + \zeta^{(b)} \ , \\
\gamma &= H_2 P(\tau\zeta + \zeta^{(b)}) + T H_2 g \ , \\
\delta &= H_2 P(\tau\sigma_\varphi + \sigma_\varphi^{(b)}) + T H_2(a\sin\theta + \cos\theta g_\varphi) \ .
\end{aligned} \tag{81}$$

We have ten linearly independent KY 1-forms and only one condition $P^2 = 1/H_2^2$ which is equivalent to $H_2'^2 = H_1^2$. For $H_2 = r$, we have $H_1 = \pm 1$, and the corresponding Killing vector fields are directly determined from (81) with $P = \pm 1/r$.

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APPENDIX A: METRIC CONSTANTS

In Section IV, the following four metric constants were defined :

$$m = L'\frac{H_0}{H_1 H_2} \ , \quad m_0 = \frac{H_0^2}{L H_1}\left(\frac{H_2}{H_0}\right)' \ , \tag{A1}$$

$$k = \frac{H_0' H_2}{H_0 H_2'} , \qquad k_0 = -\frac{H_0^2}{H_1 H_2}(\frac{H_2'}{H_1})' . \tag{A2}$$

The first relation of (A1) can be integrated to find

$$H_0' = \ell_1 H_0^{-m} H_1 H_2 , \quad L = \frac{1}{\ell_1} H_0^m , \tag{A3}$$

which is also valid for $m = 0$. The second relation of (A1) can be arranged as

$$\frac{H_2'}{H_2} = \frac{H_0'}{H_0}[m_0(\frac{H_1}{H_0'})^2 + 1] . \tag{A4}$$

The first relation of (A2) can also be integrated to find $H_0 = \ell_2 H_2^k$, where ℓ_1 and ℓ_2 are also non-zero metric constants. By combining (A4) with the first relation of (A2), we get

$$(1-k){H_0'}^2 = m_0 k H_1^2 , \tag{A5}$$

and then, by (A3) and $H_0 = \ell_2 H_2^k$, we arrive at

$$(1-k) = m_0 k (\frac{\ell_2^m}{\ell_1})^2 H_2^{2(mk-1)} . \tag{A6}$$

Thus for nonconstant H_2, we must have

$$km = 1 , \quad m = 1 + m_0(\frac{\ell_2^m}{\ell_1})^2 . \tag{A7}$$

Note that $k = 1$ if and only if $m_0 = 0$ and if and only if $m = 1$ for $H_2' \neq 0$.

APPENDIX B: CURVATURE FORMS

The curvature 2-forms $R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega^c_{\ b}$ for the considered class of spherically symmetric space-times are computed by using eqs. (2) and (3) to be as follows :

$$\begin{aligned}
R_{01} &= (S + \frac{h_0^2 + h_0' - h_0 h_1}{T^2 H_1^2})e^{01} , \quad R_{12} = S_2 e^{02} + S_1 e^{12} , \\
R_{02} &= (S + \frac{h_0 h_2}{T^2 H_1^2})e^{02} + S_2 e^{12} , \quad R_{13} = S_2 e^{03} + S_1 e^{13} , \\
R_{03} &= (S + \frac{h_0 h_2}{T^2 H_1^2})e^{03} + S_2 e^{13} , \quad R_{23} = (S_1 + \frac{H_1 P' + (H_1/H_2)^2}{T^2 H_1^2})e^{23} ,
\end{aligned} \tag{B1}$$

where

$$S = -\frac{\ddot{T}}{T H_0^2} , \quad S_1 = \frac{\dot{T}^2}{T^2 H_0^2} - \frac{P' + H_1 P^2}{T^2 H_1} , \quad S_2 = \frac{\dot{T}}{T^2}\frac{h_0}{H_0 H_1} . \tag{B2}$$

Note that $S = 0 = S_2$ when $\dot{T} = 0$. The corresponding Ricci 1-forms and curvature scalar can be found from $P_b = i_{X^a} R_{ab}$ and $\Re = i_{X^b} P_b$.

For $\dot{T} = 0 = H_0'$ and $H_2'^2 = H_1^2$ we have $S_2 = 0$ in addition to $S = 0 = S_1$. It then follows that all the curvature components vanish, and we obtain the Minkowski space-time. In the case of the static form of de Sitter space-time, that is for $\dot{T} = 0$, $H_2 = r$ and

$$H_0^2 = k_1 r^2 + k_2 \ , \ H_1^2 = \frac{1}{1 + k_3 r^2} \ , \ k_1 = k_2 k_3$$

we have $S = 0 = S_2$ and $S_1 = -k_3/T^2$ which gives the constant curvature solutions

$$R_{0j} = \frac{k_3}{T^2} e^{0j} \ , \quad R_{ij} = -\frac{k_3}{T^2} e^{ij} \ , \tag{B3}$$

where $i, j = 1, 2, 3$ and $i < j$. These two space-times are maximally symmetric with constant curvature.

For $\dot{T} = \lambda T$, $H_0' = 0$ and $H_2' = \epsilon H_1$ which indicate de Sitter space-time, we have

$$S = -\frac{\lambda^2}{H_0^2} = -S_1 \ , \quad S_2 = 0 \ .$$

These lead us again to the curvature solution given by (B3) provided that k_3/T^2 is replaced by $-\lambda^2/H_0$. For the Robertson-Walker space-time, we have

$$R_{0j} = -\frac{\ddot{T}}{T} e^{0j} \ , \quad R_{ij} = [(\frac{\dot{T}}{T})^2 + \frac{k}{T^2}] e^{ij} \ . \tag{B4}$$

Finally, for the four forms of de Sitter space-time found in Section VI, we have

$$H_0 = 1 \ , \quad H_1^2 = \frac{1}{1 - \ell r^2} \quad H_2 = r \ ,$$

and T expressions are explicitly given by equations (73). For these values we find $S_2 = 0$ and $S_1 = \ell_3 = -S$, where ℓ_3 is a constant. Thus

$$R_{0j} = -\ell_3 e^{0j} \ , \quad R_{ij} = \ell_3 e^{ij} \ . \tag{B5}$$

where we have used $\dot{T} = \epsilon(\ell_3 T^2 - \ell)^{1/2}$ found in equation (72).

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TABLE I: Metric coefficient functions and the numbers of linearly independent KY-forms of some well-known spherically symmetric space-times. For all these cases H_2 is the radial coordinate r. The numbers of the fifth column represent the dimensions $d(4,1)$ of the corresponding symmetry algebras whose common part consists of three $so(3)$ generators given by the equation (6) of the main text. The last two columns denote the numbers $d(4,2)$ and $d(4,3)$ of the linearly independent KY 2-forms and 3-forms which are explicitly calculated in the accompanying paper. The fifth rows represent three different forms of the de Sitter space-time, of which the third consists of four cases with different time evolutions given in Section VI. The numbers $d(n,p)$ for the maximally symmetric space-times, such as those of Minkowski and de Sitter, represent the upper bounds for dimension $n = 4$.

Space-time	T	H_0	H_1	$d(4,1)$	$d(4,2)$	$d(4,3)$
Schwarzschild	1	$\sqrt{1-\frac{2M}{r}}$	$\frac{1}{\sqrt{1-\frac{2M}{r}}}$	4	1	0
Reissner-Nordstrøm	1	$\sqrt{1-\frac{2M}{r}+\frac{Q^2}{r^2}}$	$\frac{1}{\sqrt{1-\frac{2M}{r}+\frac{Q^2}{r^2}}}$	4	1	0
Robertson-Walker	$T(t)$	1	$\frac{1}{\sqrt{1-kr^2}}$	6	4	1
de Sitter						
static form	1	$\sqrt{1-kr^2}$	$\frac{1}{\sqrt{1-kr^2}}$	10	10	5
usual form	$e^{(\Lambda/3)^{1/2}t}$	1	1	10	10	5
four additional forms	$\dot{T}^2 = \ell_3 T^2 - \ell$	1	$\frac{1}{\sqrt{1-\ell\, r^2}}$	10	10	5
Minkowski	1	1	1	10	10	5